J. Fluid Mech. (1983), *vol.* 137, *pp.* 273–284
Printed in Great Britain

Longshore motion due to an obliquely incident wave group

By S. C. RYRIE†
School of Mathematics, University of Bristol

(Received 18 May 1982 and in revised form 24 August 1983)

We consider longshore motion generated within the surf zone by obliquely incident breaking waves, and seek to describe the effect on such motion of variations, caused by wave grouping, in the incident longshore momentum flux. The effects of associated variations in set-up are not considered.

We use the linear long-wave equations to describe the motion resulting from the longshore momentum contained in a wave group. This consists of a succession of edge waves which disperse along the beach, and, for the example considered, an eventual steady circulation cell at the position of the wave group. We suggest that such a cell is always likely to be formed if the wave group is sufficiently localized, and that higher-modenumber edge waves are more likely to be excited.

We find timescales for the dispersal of the edge waves, and for the decay, due to bottom friction, of the circulation cell: we suggest that the latter may more generally be used, as a timescale for the effect of friction on longshore motion.

1. Introduction

Observations of wave height and surface elevation close to beaches often show consistent variations, both in time and along the beach. Munk (1949) suggested that long-period variations in amplitude, which he termed surf beat, are due to the linear superposition of two incident wavetrains of slightly differing frequencies. Bowen & Guza (1978) distinguish two effects: amplitude modulation (or grouping) of the incident waves, and the more complex phenomenon of surf beats.

Recent work has concentrated upon the role of edge waves in causing such variation. Huntley (1976) observes that edge waves can indeed make a large contribution to low-frequency energy near a coastline. Gallagher (1971) shows that nonlinear interactions between different components of the incident-wave spectrum may generate long-period edge waves near a coastline, and the experiments of Bowen & Guza (1978) suggest that surf beat is indeed strongest when conditions for the resonant growth of edge waves are satisfied. Guza & Bowen (1975) show that a regular wavetrain may be unstable to longshore perturbations as it approaches a coastline, and that this can be a mechanism for the generation of edge waves. The preferred excited waves are low-modenumber edge waves which are a subharmonic of the incident waves. If the incident waves are normal to the beach, the excited waves are standing edge waves. The experiments of Guza & Chapman (1979) confirm these effects. Foda & Mei (1981) find that normally incident, slowly modulated swell waves

† Present address: Department of Computer Studies and Mathematics, Bristol Polytechnic, Bristol BS16 1QY.

can themselves excite edge waves with longer periods than those found by Guza and Bowen.

The effect on motion within the surf zone of longshore variations of normally incident waves is studied by Bowen (1969), who considers a small sinusoidal variation in wave height along the beach, with no time dependence. He finds that the steady water motion in the surf zone takes the form of a series of circulation cells along the beach, with flow seawards out of the surf zone near points where the wave height is least. The motion is due to longshore variations in set-up, caused by the variations in breaker height: this in turn leads to a longshore pressure gradient, from an area of high waves to one of low waves, which drives the longshore velocity. This presents a possible explanation for several observed features of motion on beaches, particularly those features with a definite and regular longshore spacing, such as circulation cells and the associated rip currents. The effect of temporal variations in breaking of normally incident waves has been considered by Symonds, Huntley & Bowen (1982), who show that surf beat can, by varying the breaker position, excite standing edge waves. They suggest that obliquely incident waves may similarly generate edge waves.

When waves are obliquely incident, the effect of amplitude variations on surf zone motion is twofold, being due (*a*) to variations in set-up, and (*b*) to variations in the flux of longshore momentum incident on the surf zone. The former exists regardless of obliqueness, while the latter is caused only by obliquely incident waves. In this paper we consider only the effect of (*b*) on longshore motion in the surf zone. The motion resulting from this should be seen as being an additional effect of wave grouping, caused only by obliquely incident waves, for which, nevertheless, the two effects (*a*) and (*b*) must occur together. We shall, in considering only the latter, thereby ignore any interaction between the effects of variations in set-up and in longshore momentum flux. We shall use linearized dynamics, therefore also neglecting interaction between longshore motion and any changes in onshore motion and water depth. The results found here are, accordingly, a guide to the physical features likely in practice to be found, in addition to the circulation cells found by Bowen (1969). We consider, in §4, the circumstances in which the results we find may be observable.

For simplicity, we shall consider only the effect of an individual group of obliquely incident waves. Since we are using linearized dynamics, the full effect of wave groupings would appear as a superposition, along the beach, of the results found below. For our purposes, since we are considering only the effect of the resulting changes in incident momentum flux, we may model the arrival of such a group as the sudden deposition within the surf zone of a finite 'packet' of longshore momentum with some given spatial distribution. This represents the local increase in incident longshore momentum flux due to increased wave height: Longuet-Higgins (1970) shows how the wave-induced longshore stress, which drives longshore motion, is related to incident wave height. In order for the arrival of the longshore momentum correctly to be described as impulsive, the duration of the wave group and the time taken for the incident longshore momentum to spread shorewards must both be less than the timescales of changes in the subsequent motion, as discussed later.

We shall consider, in §§2 and 3 respectively, the transient and the eventual steady motion caused by a given initial distribution of longshore momentum. We omit, in these sections, any representation of bottom friction. Foda & Mei (1981) find, in similar circumstances, that the inclusion of bottom friction affects time-dependent motion only quantitatively. However, the 'steady' motion discussed in §3 may be expected to decay, owing to friction, and in §4 we estimate a timescale for changes in longshore motion when the incident longshore momentum flux (which is determined by the incident waves) is insufficient to match frictional dissipation (determined by motion within the surf zone). Since these are the circumstances following the arrival of a wave group, this gives a timescale for the decay of the motion described in §3. We also, in §4, consider the circumstances in which the timescale for the dispersal of the transient edge waves, described in §2, may be sufficiently long for them to be observable.

2. Transient motion

We suppose that, at some initial moment, a finite amount of longshore momentum is instantaneously deposited in the surf zone. We define it by specifying an initial distribution of longshore velocity, which may be rotational if we assume it to have been introduced by an obliquely incident group of waves. We use linear theory, so that we may neglect interactions with any other surf-zone motion, and in particular with changes in onshore momentum caused by the arrival of such a wave group. We also, for simplicity, neglect friction, but note that any long-lasting motion found as a solution may be expected to decay, because of friction, on a timescale determined by the incident waves; we discuss this further in §4.

A full description of the linearized long-wave equations is given by Peregrine (1972), who notes that even in circumstances where they are not strictly applicable, as is the case here, they give a qualitative guide to what might happen; it is in this spirit that we adopt them here. They may be written

$$\eta_t + h_0(x)\,(u_x + v_y) + h_0'(x)\,u = 0, \tag{1a}$$

$$u_t + g\eta_x = 0, \tag{1b}$$

$$v_t + g\eta_y = 0, \tag{1c}$$

where x and y are coordinates measured respectively seawards and along the beach; $x = 0$ represents the shoreline; $h_0(x)$ is the undisturbed water depth for a beach with no longshore variations in topography; $\eta(x, y, t)$ is the surface displacement, so that the total depth is

$$h(x, y, t) = h_0(x) + \eta(x, y, t),$$

and $u(x, y, t)$ and $v(x, y, t)$ are respectively x- and y-components of depth-averaged water velocity.

We seek solutions of (1) satisfying a given initial velocity distribution

$$v(x, y, 0) = v_0(x, y), \quad u(x, y, 0) = 0 \tag{2}$$

with boundary conditions

$$u(0, y, t) = 0, \quad v(0, y, t)\ \text{bounded}. \tag{3}$$

This is similar to the initial-value problem studied by Whitham (1979), who finds solutions to the one-dimensional problem ($v \equiv 0$ in (1)) for a given initial surface displacement. His solutions are a linear superposition of edge-wave modes; we shall find solutions in a similar manner.

We firstly combine (1) so as to eliminate η and u, to find

$$v_{ttt} - gh_0(v_{xx} + v_{yy})_t - gh_0' v_{xt} = 0. \tag{4}$$

An alternative procedure is to eliminate u and v, finding an equation only in η, as by Whitham (1979). However, this does not so readily show the existence of the steady solutions, described below, for which η is identically zero.

One solution of (4) is clearly given by $v_t = 0$: this leads to the time-independent solutions already mentioned. For the moment, we assume that $v_t \neq 0$, and seek solutions of the form

$$v(x, y, t) = \mathrm{Re}\{q(x, y)\exp(\mathrm{i}\omega t)\} \quad (\omega \neq 0). \tag{5}$$

We also take the beach to be plane and regular with slope s, so that

$$h_0(x) = sx.$$

Using (5) in (4) and taking a Fourier transform in space, so that

$$Q(x, l) = \int_{-\infty}^{\infty} q(x, y)\,\mathrm{e}^{\mathrm{i}ly}\,\mathrm{d}y, \tag{6}$$

we find

$$Q'' + \frac{1}{x}Q' + \left(\frac{\omega^2}{gsx} - l^2\right)Q = 0, \tag{7}$$

where a prime denotes differentiation with respect to x. Whitham (1979) finds the solutions bounded at $x = 0$ and ∞ as

$$Q_n(x, l) = \mathrm{e}^{-|l|x} L_n(2|l|\,x) \quad (n = 0, 1, 2, \ldots), \tag{8}$$

where the L_n are Laguerre polynomials

$$L_n(z) = \frac{\mathrm{e}^z}{(n!)^2}\frac{\mathrm{d}^n}{\mathrm{d}z^n}(z^n\,\mathrm{e}^{-z}),$$

for values of ω given by

$$\omega^2 = sg(2n+1)\,|l|. \tag{9}$$

The full solution of (7) is therefore a superposition of the eigenfunctions (8):

$$Q(x, l) = \sum_{n=0}^{\infty} A_n(l)\,L_n(2|l|\,x)\,\mathrm{e}^{-|l|x},$$

and so

$$v(x, y, t) = \frac{1}{2\pi}\mathrm{Re}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty} A_n(l)\,L_n(2|l|\,x)\,\mathrm{e}^{-|l|x-\mathrm{i}(ly-\omega t)}\,\mathrm{d}l, \tag{10}$$

where the $A_n(l)$ are chosen to satisfy the initial conditions (2): the appropriate inversion theorem as used by Whitham (1979) gives

$$A_n(l) = 2\int_0^{\infty}\int_{-\infty}^{\infty} v_0(x, y)\,L_n(2|l|\,x)\,|l|\,\mathrm{e}^{-|l|x+\mathrm{i}ly}\,\mathrm{d}y\,\mathrm{d}x. \tag{11}$$

Equation (1c) gives the surface displacement as

$$\eta(x, y, t) = \frac{1}{2\pi g}\mathrm{Re}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty} A_n(l)\,L_n(2|l|\,x)\frac{\omega}{l}\mathrm{e}^{-|l|x-\mathrm{i}(ly-\omega t)}\,\mathrm{d}l, \tag{12}$$

to which an arbitrary function of x and t may be added, allowing for the onshore–offshore motion.

These solutions are superpositions of shallow-water edge-wave modes, which travel along the beach, with amplitude decreasing seawards. We have therefore described a mechanism for the excitation of edge waves, due to longshore variations in the amplitude of obliquely incident breaking waves. Which modenumbers actually occur depends upon particular circumstances. However, the edge waves described here are excited by an input of longshore momentum into the surf zone, and longshore velocity normally increases with distance from the shore to a maximum at some distance offshore. Thus a particular modenumber is more likely to be excited in this way if its energy is not too concentrated near the shoreline. This suggests that higher-numbered modes are more favoured, since, for a given frequency, the amplitude of the edge wave of a particular modenumber decreases with distance from the shore less rapidly than does that of an edge wave of a lower modenumber.

Since edge waves are dispersive, and, in the present problem, there is no driving for the motion after the initial disturbance, we may expect the amplitude of the edge-wave solutions found here to decrease with time as the waves disperse along the beach. In order to show this, we consider the behaviour of the solution (12) for large values of t.

Since we are considering a linearized problem, we may write the initial distribution as

$$v_0(x,y) = \int_{-\infty}^{\infty}\int_0^{\infty} v_0(x_0,y_0)\,\delta(x-x_0)\,\delta(y-y_0)\,\mathrm{d}x_0\,\mathrm{d}y_0 \tag{13}$$

and find the solution for the surface elevation as a superposition of Green functions, i.e. as

$$\eta(x,y,t) = \int_{-\infty}^{\infty}\int_0^{\infty} v_0(x_0,y_0)\,G(x,y,t;x_0,y_0)\,\mathrm{d}x_0\,\mathrm{d}y_0,$$

where $G(x,y,t;x_0,y_0)$ is the solution of the initial-value problem for which

$$v_0(x,y) = V_0\,\delta(x-x_0)\,\delta(y-y_0). \tag{14}$$

For this initial distribution the A_n are obtained from (11) as

$$A_n(l) = 2V_0 L_n(2|l|\,x_0)\,|l|\,\mathrm{e}^{-|l|x_0+\mathrm{i}ly_0},$$

and we find the solution from (12) as

$$G(x,y,t;x_0,y_0) = \frac{V_0}{\pi g}\sum_{n=0}^{\infty}\alpha_n\int_0^{\infty} l^{\frac{1}{2}}L_n(2lx)\,L_n(2lx_0)\,\mathrm{e}^{-l(x+x_0)}$$
$$\times\{\cos(l(y-y_0)-\omega t)-\cos(l(y-y_0)+\omega t)\}\,\mathrm{d}l, \tag{15}$$

where $\alpha_n = (sg(2n+1))^{\frac{1}{2}}$, so that $\omega = \alpha_n|l|^{\frac{1}{2}}$, from the edge-wave dispersion relation (9).

A stationary-phase asymptotic expansion of (15) gives

$$G(x,y,t;x_0,y_0) \sim \frac{2V_0}{\pi g}\sum_{n=0}^{\infty}\frac{\omega_0^2}{\alpha_n^2}L_n\left(\frac{2\omega_0^2}{\alpha_n^2}x\right)L_n\left(\frac{2\omega_0^2}{\alpha_n^2}x_0\right)$$
$$\times\exp\left[-\frac{\omega_0^2}{\alpha_n^2}(x+x_0)\right]\int_0^{\infty}\cos\left\{[\omega^2-2\omega_0\omega]\frac{y-y_0}{\alpha_n^2}\right\}\mathrm{d}\omega$$

as $t\to\infty$, where $\omega_0 = t\alpha_n^2/2|y-y_0|$. Using Gradshteyn & Ryzhik (1965, §(3.693)) to evaluate the integral, we obtain

$$G(x,y,t;x_0,y_0) \sim \frac{V_0}{4(2\pi)^{\frac{1}{2}}g}\frac{t^2}{|y-y_0|^{\frac{5}{2}}}\sum_{n=0}^{\infty}\alpha_n^3 L_n\left(\frac{\alpha_n^2t^2x}{2(y-y_0)^2}\right)L_n\left(\frac{\alpha_n^2t^2x_0}{2(y-y_0)^2}\right)$$
$$\times\left[\cos\frac{\alpha_n^2t^2}{4(y-y_0)}+\sin\frac{\alpha_n^2t^2}{4(y-y_0)}\right]\exp\left(-\frac{\alpha_n^2(x+x_0)\,t^2}{4(y-y_0)^2}\right).$$

The amplitude of the nth mode is therefore

$$O\left\{t^{4n+2}\exp\left[-\tfrac{1}{4}sg(2n+1)\frac{x+x_0}{(y-y_0)^2}t^2\right]\right\}$$

as $t\to\infty$.

If we take the surf-zone width to be a typical value for $x+x_0$, this gives a timescale for the dispersion of a particular edge-wave mode at a point a distance $y-y_0$ from the original disturbance as

$$\tau_n = \frac{2(y-y_0)}{(gh_{\mathrm{B}}(2n+1))^{\frac{1}{2}}},$$

where h_{B} is the depth at breaking. Thus, at any given point along the beach, the rate of dispersion increases with modenumber. Since the zero-modenumber component of the motion disperses most slowly, we define a timescale for the dispersion of the edge-wave motion as

$$\tau_{\mathrm{e}} = \frac{2(y-y_0)}{(gh_{\mathrm{B}})^{\frac{1}{2}}}. \tag{16}$$

3. Steady motion

In the original initial-value problem there is a distribution of vorticity in the surf zone. Since the linear shallow-water equations (1) conserve vertical vorticity, this distribution must remain even after the edge waves generated by the initial disturbance have dispersed. There must therefore, if linear theory is used, and bottom friction is neglected, be left in the surf zone a residual steady velocity distribution. Upon setting $\partial/\partial t = 0$ in (1) to give a steady solution, we find that the only non-trivial equation is

$$h_0(u_x+v_y)+h_0' u = 0, \tag{17}$$

where u and v are now functions of x and y which preserve the original vorticity distribution, so that

$$v_x - u_y = \frac{\partial v_0}{\partial x} = \zeta(x,y) \quad \text{say}, \tag{18}$$

together with the conditions (3). We now seek a solution of (17) and (18) in terms of a mass-transport stream function $\psi(x,y)$ defined by

$$\psi_x = xv, \quad \psi_y = -xu, \tag{19}$$

for a plane beach with $h_0 = sx$. Equation (19) satisfies (17) identically, and (18) becomes

$$x(\psi_{xx}+\psi_{yy})-\psi_x = x^2\zeta(x,y).$$

Defining a Fourier transform by

$$\bar{\psi}(x,l) = \int_{-\infty}^{\infty} \psi(x,y)\,\mathrm{e}^{\mathrm{i}ly}\,\mathrm{d}y, \tag{20}$$

we find

$$z^2\bar{\psi}_{zz} - z\bar{\psi}_z - z^2\bar{\psi} = \frac{z^3}{l^3}\bar{\zeta}\left(\frac{z}{l},l\right), \tag{21}$$

where we have changed the independent variable to $z = lx$. The solutions to the homogeneous form of (21) are

$$\bar{\psi}(z,l) = z[A(l)\,I_1(z)+B(l)\,K_1(z)] \tag{22}$$

for arbitrary $A(l), B(l)$; a solution to the inhomogeneous equation is

$$\bar{\psi}(z,l) = \frac{z}{l^3}\left(I_1(z)\int zK_1(z)\,\bar{\zeta}\left(\frac{z}{l},l\right)\mathrm{d}z - K_1(z)\int zI_1(z)\,\bar{\zeta}\left(\frac{z}{l},l\right)\mathrm{d}z\right) \tag{23}$$

(Kamke 1948 §A (24.2)), where I_1 and K_1 are modified Bessel functions.

The full solution is the sum of (22) and (23). After satisfying the boundary conditions, we find this to be

$$\bar{\psi}(x,l) = -\frac{x}{l^2}\left\{I_1(lx)\int_{lx}^{\infty} zK_1(z)\,\bar{\zeta}\left(\frac{z}{l},l\right)\mathrm{d}z + K_1(lx)\int_0^{lx} zI_1(z)\,\bar{\zeta}\left(\frac{z}{l},l\right)\mathrm{d}z\right\}$$

$$= \frac{x}{l}\left\{K_1(lx)\int_0^{lx} zI_0(z)\,\bar{v}_0\left(\frac{z}{l},l\right)\mathrm{d}z - I_1(lx)\int_{lx}^{\infty} zK_0(z)\,\bar{v}_0\left(\frac{z}{l},l\right)\mathrm{d}z\right\}, \qquad (24)$$

after integrating by parts, and assuming zero longshore velocity at the shoreline.

We shall show, for illustration, an example of a solution with an initial longshore velocity distribution

$$v_0(x,y) = V_0 f(x)\,g(y), \qquad (25)$$

where we take

$$f(x) = \frac{x}{x_0}\mathrm{e}^{-x/x_0}, \quad g(y) = \mathrm{e}^{-(y/y_0)^2} \qquad (26)$$

for some constants x_0, y_0 and V_0. This form of $f(x)$ gives a distribution of longshore current across the surf zone which is qualitatively similar to those often found in practice.

The form of $g(y)$ in (26) is chosen to give a distribution sharply peaked at $y = 0$, and close to zero far along the beach, as is appropriate in considering the effect of a localized wave group. The corresponding solution of (21) is

$$\bar{\psi}(x,l) = -\frac{V_0\pi^{\frac{1}{2}}y_0}{x_0}\frac{x}{l^2}\{K_1(lx)\,R_I(x_1 l) - I_1(lx)\,R_K(x_1 l)\}\exp\left(-\tfrac{1}{4}y_0^2 l^2\right);$$

by inverting the Fourier transform, we find the stream function to be

$$\psi(x,y) = -\frac{V_0 y_0}{\pi^{\frac{1}{2}}}\frac{x}{x_0}\int_0^{\infty}\frac{1}{l^2}[I_1(lx)\,R_K(x,l) - K_1(lx)\,R_I(x,l)]\cos(ly)\exp\left(-\tfrac{1}{4}y_0^2 l^2\right)\mathrm{d}l, \qquad (27)$$

where

$$R_K(x,l) = -\int_{lx}^{\infty} z\left(1-\frac{z}{lx_0}\right)K_1(z)\exp\left(\frac{-z}{lx_0}\right)\mathrm{d}z,$$

$$R_I(x,l) = \int_0^{lx} z\left(1-\frac{z}{lx_0}\right)I_1(z)\exp\left(\frac{-z}{lx_0}\right)\mathrm{d}z.$$

Standard numerical routines were used to evaluate all the integrals. The semi-infinite integral in (27) was calculated by truncation of the range at a finite value $l_{\max}$ beyond which the integrand is sufficiently small to be considered negligible: at $l_{\max} = 20$ its value was always less than machine tolerance, and this limit was used.

In figure 1 we show contours of $\psi(x,y)$ calculated from (27) and of the vorticity $\zeta(x,y)$. The values $V_0 = 1$, $x_0 = y_0 = 1$ were used. A circulation cell is seen, in the region near the maximum of the original distribution V_0: shoreward of its centre, the direction of the longshore motion is reversed from that of the initial motion. On each side, along the beach, of the circulation cell, there is a current, flowing seaward on one side and shoreward on the other. The angle that this flow makes with the shoreline depends on the dimensions of the problem: if the longshore scale y_0 is much less than the offshore scale x_0, which is a measure of the surf-zone width, this current flows almost directly seaward or shoreward close to the circulation cell.

In the calculation of particle velocities, the aspect ratio x_0/y_0 is a free parameter. We show in figure 2 profiles of longshore velocity $v(x,y)$, for $x_0/y_0 = 1$, in order to compare the original distribution $v_0(x,y)$ with the final steady distribution derived

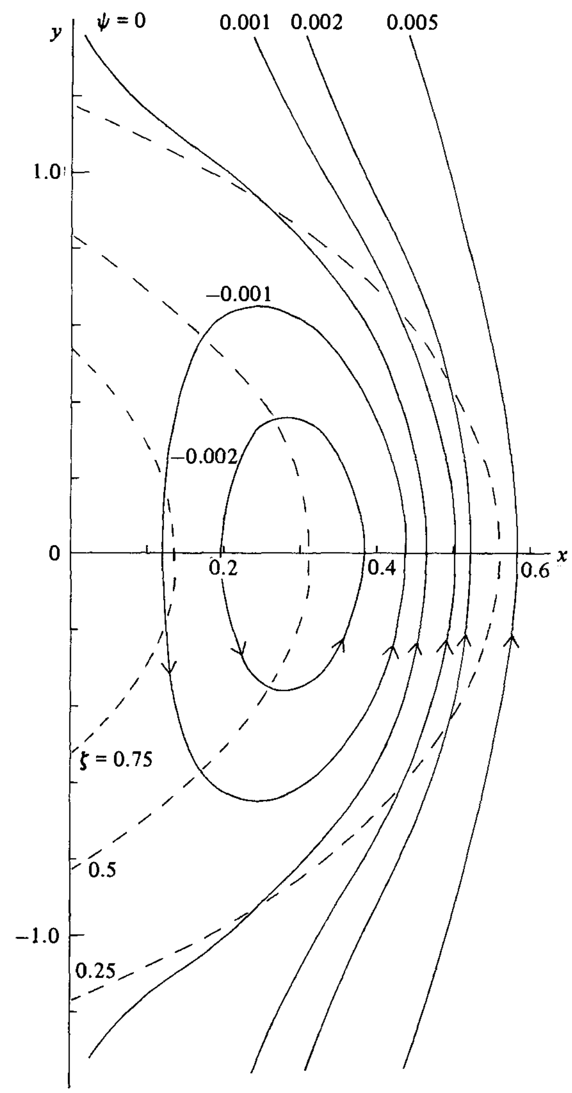

FIGURE 1. Streamlines and contours of vorticity for steady motion on a plane beach. The shoreline is at $x = 0$. ——, ψ = constant; --------, ζ = constant.

from (27). Comparisons of these profiles are shown at various values of y, within the region of interest, i.e. within the circulation cell. There are significant changes in v within this region, of the order of the original magnitude of v_0.

Although we have found that the particular initial conditions of (25) and (26) eventually give rise to a circulation cell, details of the flow resulting from any given initial conditions are obscured by the complexity of (24). It would, however, be of particular interest to know under what conditions on the initial velocity a circulation cell might result.

We suppose that the presence of a cell is indicated by a negative value of longshore velocity at the origin, $x = 0$ and $y = 0$, using (24) to find this value. By inverting the Fourier transform, and finding $v(x, y)$ using (19), we obtain

$$v(0,0) = -\frac{1}{\pi}\int_0^\infty \int_0^\infty l^2 x K_0(lx)\, \bar{v}_0(x,l)\, \mathrm{d}l\, \mathrm{d}x.$$

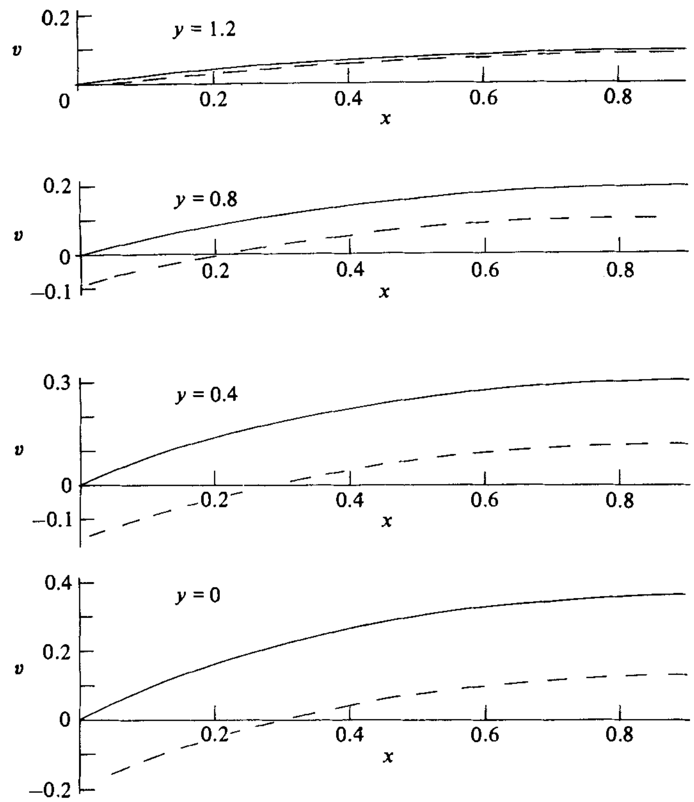

FIGURE 2. Longshore velocity versus distance from shoreline, at four stations along the beach, showing ——, the initial longshore velocity $v_0(x, y)$, from (25) and (26), and – – – – – – – – –, the corresponding eventual steady longshore velocity, with a circulation cell present.

A sufficient condition for $v(0,0) < 0$ is therefore that

$$\bar{v}_0(x, l) \geqslant 0 \quad \forall x \geqslant 0, l \geqslant 0.$$

This indicates that, if $v_0(x, y)$ is sharply peaked near $y = 0$ for all values of x, that is that the initial motion is concentrated near a single point on the beach, then a circulation cell will be formed.

A succession of surf beats at intervals along a shore may be expected to produce a series of such circulation cells, with a pattern broadly similar to that found by Bowen (1969). The mechanisms for their generation are, however, different: those of Bowen are due to the variations in set-up associated with changes in breaker height, and may be caused by normally incident waves; those found here result from the longshore momentum associated with obliquely incident waves.

4. Timescale

Whether either the transition edge waves or the residual velocity distribution already described is in practice observed depends upon the timescale of their duration. The edge waves disperse with a timescale τ_e given by (16), while the 'steady' velocity distribution of §3 decays with time owing to bottom friction, whose effect has so far been neglected. We shall, however, make an estimate of the timescale of this decay.

A steady longshore current, when such exists, is a balance between the incident longshore momentum flux and friction, which depends on the current itself. Thus, if longshore motion is suddenly increased owing to the arrival of a wave group, the rate of dissipation due to friction is also increased. If, however, as we suppose here, the subsequent incident longshore momentum flux does not match the increased frictional dissipation, the effect of friction will be to decrease the overall longshore momentum until the abovementioned balance again exists. We seek a timescale for this process.

Within the surf zone we again adopt the linearized shallow-water equations, with the inclusion of a term representing bottom friction: we use a quadratic friction law, under which the bottom friction stress is

$$\boldsymbol{B} = C\rho\boldsymbol{u}|\boldsymbol{u}|,$$

where $\boldsymbol{u}$ is local fluid velocity and C is a dimensionless constant. If we suppose that the longshore velocity is always much less than the onshore velocity, we may write the longshore momentum equation as

$$v_t + gh_y + C\frac{|u|v}{h} = 0. \tag{28}$$

A consideration of the first and last terms of (28) shows that a characteristic timescale for changes in longshore momentum due to bottom friction is the value of

$$\tau_f = \frac{h}{CU_m}, \tag{29}$$

where $U_m = \overline{|u|}$. We therefore use τ_f as a timescale for the decay of the steady motion described in §3.

Equation (28) may more formally be used (Ryrie 1981) to show that τ_f is the timescale on which friction allows longshore motion to react to a change in the incident momentum flux. The same timescale is found by Foda & Mei (1981) for the rate of change, due to bottom friction, of energy in long waves in the surf zone. We suggest therefore that τ_f, as defined by (29), may more generally be used as a timescale for the effect of friction acting on longshore motion.

A further timescale in the problem is the period of changes in the amplitude of the incident waves, which may be due to modulations in the offshore wavetrain, or to edge waves: we use τ_s to denote this timescale. Of interest is the ratio τ_e/τ_s, whose value indicates whether edge waves excited by the arrival of a wave group, as described in §2, are likely to be observable between successive wave groups.

If wave groups are the result of modulations in the incident wavetrain, caused, perhaps, by interference between waves of slightly differing frequencies, then the longshore spacing of such groups is $L = (c_B/\sin\theta_B)\,\tau_s$, since wave crests travel along the shore with speed $c_B/\sin\theta_B$. If we take $\frac{1}{2}L$ to be a typical value of y in (16), and $c_B \approx (gh_B)^{\frac{1}{2}}$ then we find

$$\frac{\tau_e}{\tau_s} \approx \frac{1}{\sin\theta_B}.$$

Thus, if the angle of incidence is small, edge waves excited as in §2 disperse only slowly, but, on the other hand, their total energy is also small, and is zero if θ_B is zero. Note, however, that in the analysis of §2 we consider the effect of a wave group arriving at only one point on a beach at a given time: this is no longer strictly appropriate when considering a wave group moving along the beach with the incident waves.

A wave group may also be due to edge waves generated, for instance, in the way described by Guza & Bowen (1975). In this case we may take L to be the wavelength of any such edge waves. If we also take x_0 to be the surf-zone width, so that $h_B = sx_0$, and again use $y = \frac{1}{2}L$ in (16), we find, after using the edge-wave dispersion relation,

$$\frac{\tau_e}{\tau_s} = \left(\frac{L}{2\pi x_0}\right)^{\frac{1}{2}},$$

independently of edge-wave modenumber. Thus, if the longshore spacing L is at least as great as the surf-zone width, we may expect any edge waves that are excited as described in §2 to be observable, in between the occurrence of wave groups, and before they disperse.

It is of interest to apply the above discussion to a practical example in order to find possible values of the timescales described. As part of the field observations described by Packwood (1980) on Putborough Sands in N. Devon, a record was made of the breaking position of waves arriving at one point on the beach. This record was analysed for a period of 24 minutes. Since breaker height is often approximately proportional to depth at breaking and hence to distance from the shore, this provides a rough record of amplitude at breaking. The mean surf-zone width at the time was about 70 m, the mean wave period was about 7 s, and the beach slope was almost constant at about 1:60.

Packwood finds that the breaker amplitudes appear to lie in the range 0.3–2 m. Although their variation with time is not regular, there is evidence of irregular variations with a timescale in the range 30–50 s. If we take a value for this timescale as $\tau_s \approx 40$ s, and use $L \approx 42$ m as the wavelength of an edge wave with this period, we find $\tau_e/\tau_s \approx 0.3$; the same value holds for all modenumbers. If the incident waves had been approaching at an angle to the shoreline, we might expect, on the basis of this estimate for the particular wave record used, that edge waves excited as described in §2 may make an appreciable contribution to longshore motion.

The timescale τ_f of frictional decay may be estimated by the averaging process used to obtain (8). We take $\bar{h} \approx 1$ m as a typical depth in the surf zone: $u_m \approx \frac{1}{2}\alpha(g\bar{h})^{\frac{1}{2}} = 0.6$ m/s, where $\alpha \approx 0.4$ is the ratio of amplitude to depth, and $C \approx 0.01$ as the friction coefficient. Equation (29) gives $\tau_f \approx 3$ min, so that, in this particular case, friction acts only slowly compared to the wave period and the timescale of edge-wave dispersion.†

5. Concluding remarks

In considering unsteady longshore motion, we have discussed both its response to fluctuations in the breaking waves and its transition from one steady state to another.

The former problem remains of considerable interest. We have carried out only a simple treatment of the effects of fluctuations in the incident waves on longshore motion. However, we have been able to indicate some likely results. We have shown that grouping of an obliquely incident wavetrain may excite edge waves, and have suggested that high modenumbers may, in practice, be the most likely to be generated. We have also shown the possible occurrence of circulation cells caused by variations in incident longshore momentum flux.

† The validity of this conclusion depends strongly on the value used of the friction coefficient C. A referee has observed that, in the presence of a wave boundary layer, C may be sufficiently large ($C \approx 0.05$) to make τ_f, in this example, comparable to τ_s in magnitude.

We have also derived a timescale on which weak longshore motion changes from one steady periodic motion to another. The existence of such a timescale may be significant in practical applications of longshore current predictions, in which it is often implicitly assumed that an instantaneous balance is made between driving stress and friction.

I am grateful to Dr D. H. Peregrine for many helpful discussions, and to referees for their comments. The financial assistance of the Science and Engineering Research Council is gratefully acknowledged.

REFERENCES

BOWEN, A. J. 1969 Rip currents, 1: theoretical investigations. *J. Geophys. Res.* **74**, 5467–5478.

BOWEN, A. J. & GUZA, R. T. 1978 Edge waves and surf beat. *J. Geophys. Res.* **83**, 1913–1920.

FODA, M. A. & MEI, C. C. 1981 Nonlinear excitation of long-trapped waves by a group of shore swells. *J. Fluid Mech.* **111**, 319–345.

GALLAGHER, B. 1971 Generation of surf beat by nonlinear wave interactions. *J. Fluid Mech.* **49**, 1–20.

GRADSHTEYN, I. S. & RYZHIK, I. M. 1965 *Table of Integrals, Series, and Products.* Academic.

GUZA, R. T. & BOWEN, A. J. 1975 The resonant instabilities of long waves obliquely incident on a beach. *J. Geophys. Res.* **80**, 4529–4534.

GUZA, R. T. & CHAPMAN, D. C. 1979 Experimental study of the instabilities of waves obliquely incident on a beach. *J. Fluid Mech.* **95**, 199–208.

HUNTLEY, D. A. 1976 Long period waves on a natural beach. *J. Geophys. Res.* **81**, 6441–6449.

LONGUET-HIGGINS, M. S. 1970 Longshore currents generated by obliquely incident sea waves, 1 and 2. *J. Geophys. Res.* **75**, 6778–6789 and 6790–6801.

MUNK, W. H. 1949 Surf beats. *Trans. Am. Geophys. Union* **30**, 849–854.

PACKWOOD, A. R. 1980 Surf and run-up on a beach. Ph.D. thesis, University of Bristol.

PEREGRINE, D. H. 1972 Equations for water waves and the approximations behind them. In *Waves on Beaches* (ed. R. E. Meyer), pp. 95–121. Academic.

RYRIE, S. C. 1981 Waves obliquely incident on a beach. Ph.D. thesis, University of Bristol.

SYMONDS, G., HUNTLEY, D. A. & BOWEN, A. J. 1982 Two-dimensional surf beat: long wave generation by a time-varying breakpoint. *J. Geophys. Res.* **87**, 492–498.

WHITHAM, G. B. 1979 *Lectures on Wave Propagation.* Springer for Tata Institute, Bombay.